

Lesson 012

Expected Values

Friday, October 6

**If you bet \$100 on a coin toss,
what do you expect to happen?**

If you flip a coin 100 times, how many heads do you expect to observe?

Expected Value

- The **expected value** is the average of a **probability distribution**.
- Also referred to as the **expectation**.
- For a random variable, X , this is denoted $E[X]$.

Connection to Sample Mean

$\{1, 2, 2, 3, 7\}$

$$\bar{x} = \frac{1 + 2 + 2 + 3 + 7}{5} = \frac{15}{5} = 3$$

Connection to Sample Mean

$$\bar{x} = \frac{1 + 2 + 2 + 3 + 7}{5} = \frac{15}{5} = 3$$

$$\bar{x} = \frac{1}{5}(1) + \frac{2}{5}(2) + \frac{1}{5}(3) + \frac{1}{5}(7)$$

Connection to Sample Mean

$\{1, 2, 2, 3, 7\}$

$p(x) =$

$$\begin{cases} \frac{1}{5} & x = 1 \\ \frac{2}{5} & x = 2 \\ \frac{1}{5} & x = 3 \\ \frac{1}{5} & x = 7 \\ 0 & \text{otherwise} \end{cases}$$

Connection to Sample Mean

$$X \sim p(x) \implies E[X] = 3$$

Connection to Sample Mean

$$\begin{aligned} E[X] &= (1)P(X = 1) + (2)P(X = 2) + \\ &\quad (3)P(X = 3) + (7)P(X = 7) \\ &= (1)p(1) + (2)p(2) + \\ &\quad (3)p(3) + (7)p(7) \end{aligned}$$

Connection to Sample Mean

$$\begin{aligned} E[X] &= \frac{1}{5}(1) + \frac{2}{5}(2) + \frac{1}{5}(3) + \frac{1}{5}(7) \\ &= 3 \end{aligned}$$

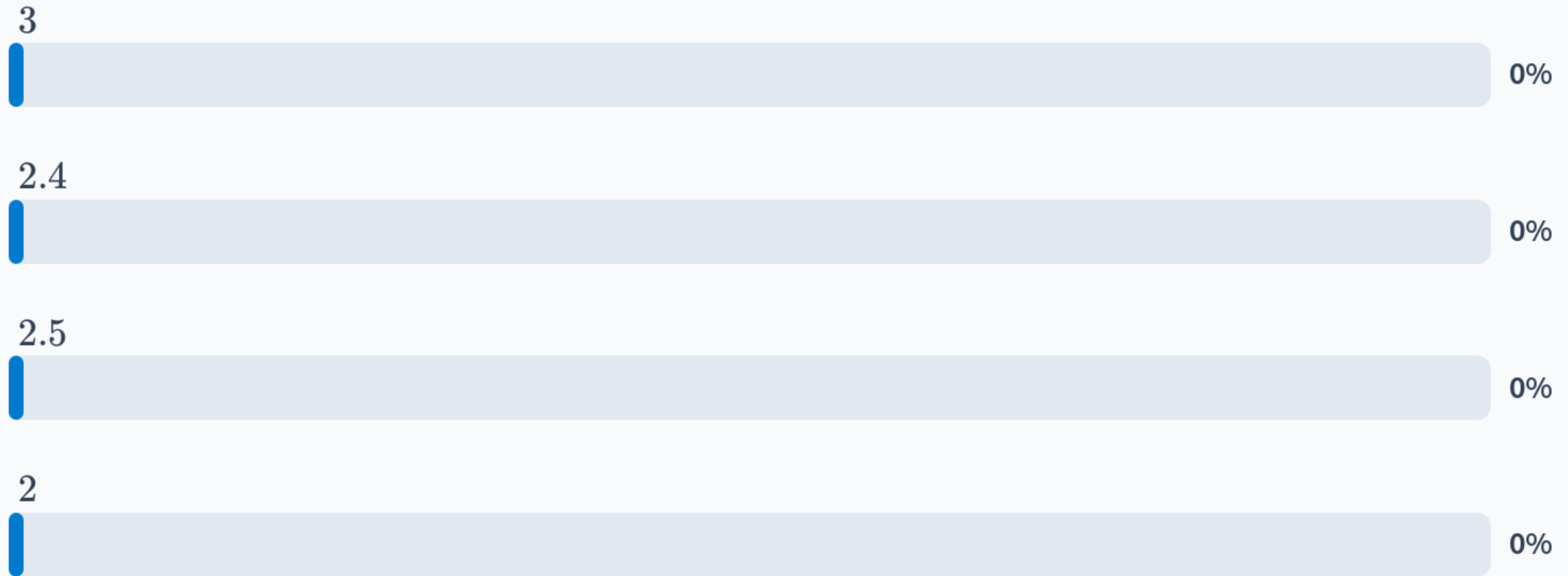
Expected Value

- For $X \sim p(x)$ we get

$$E[X] = \sum_i x_i p(x_i)$$

- In general $E[X] \neq \bar{x}$.

A particular illness is treated with a course of treatments, until symptoms subside. Cases require 1, 2, 3, or 4 rounds of treatment in 20%, 30%, 40%, and 10% of situations, respectively. How many treatments are expected for a newly diagnosed individual?



What is the expected value of the Bernoulli distribution? $p(x) \begin{cases} p & x = 1 \\ 1 - p & x = 0 \\ 0 & \text{otherwise} \end{cases}$

p
0%

$1 - p$
0%

1
0%

$\frac{p}{1-p}$
0%

Properties of Expectations

- For $X \sim p(x)$ and a function h , we get

$$E[h(X)] = \sum_i h(x_i)p(x_i)$$

- In general $E[h(X)] \neq h(E[X])$.

Expectations of Linear Functions

- If $h(X) = a + bX$, then

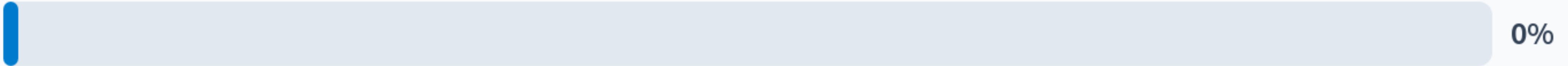
$$E[h(X)] = a + bE[X]$$

- Example: T is temperature in Celsius, $F = 1.8T + 32$. Then

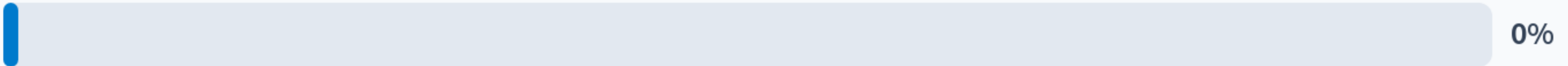
$$E[F] = 32 + 1.8E[T]$$

The expected weight of a part produced in a manufacturing plant is $E[X] = 4.5\text{kg}$. The manufacturer begins dealing with a customer who wants the weight in pounds ($1\text{kg}=2.2\text{lbs}$). What is the expected weight in pounds?

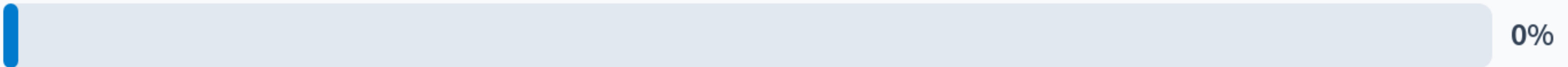
9.9



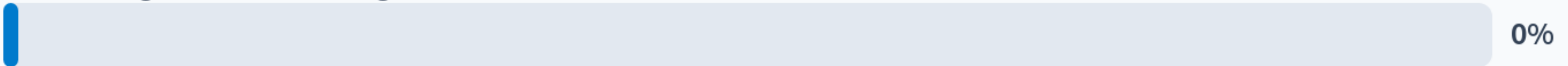
2.05



6.7



Not enough information is given.



A die manufacturer is considering the variability in their process. They find that the expected length of a side on dice that they produce is 1.6cm. What is the expected volume of the die, assuming that the process always produces exactly cubic dice?

$$1.6^3 = 4.096$$

0%

$$3 \times 1.6 = 4.8$$

0%

$$1.6^2 = 2.56$$

0%

Not enough information is provided.

0%

Variance

- Recall that the **sample variance** was the **sample average** of squared deviations from the mean.
- The **population variance** or **variance of a distribution** is the analogous quantity, with expectations.

Variance

- For $X \sim p(x)$ we get

$$\text{var}(X) = E \left[(X - E[X])^2 \right]$$

- This is $E[h(X)]$ for $h(X) = (X - E[X])^2$.

$$\text{var}(X) = E[X^2] - E[X]^2$$

A particular illness is treated with a course of treatments, until symptoms subside. Cases require 1, 2, 3, or 4 rounds of treatment in 20%, 30%, 40%, and 10% of situations, respectively. What is the variance in the number of treatments required?

$$6.6 - 2.5^2 = 0.35$$

0%

2.4

0%

$$6.6 - 2.4^2 = 0.84.$$

0%

6.6

0%

What is the variance of the Bernoulli distribution? $p(x) \begin{cases} p & x = 1 \\ 1 - p & x = 0 \\ 0 & \text{otherwise} \end{cases}$

p
0%

$p(1 - p)$
0%

1
0%

p^2
0%

Properties of Variance

- For $X \sim p(x)$ and a function h , we get

$$\text{var}\{h(X)\} = E \left[(h(X) - E[h(X)])^2 \right]$$

- In general $\text{var}\{h(X)\} \neq h(\text{var}(X))$.

Variance of Linear Functions

- If $h(X) = a + bX$, then

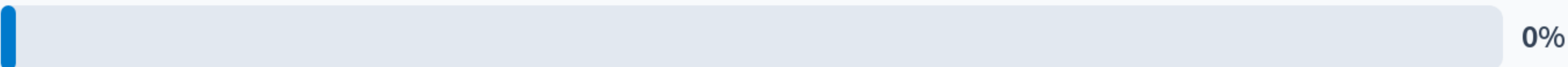
$$\text{var}\{h(X)\} = b^2 \text{var}(X)$$

- Example: T is temperature in Celsius, $F = 1.8T + 32$. Then

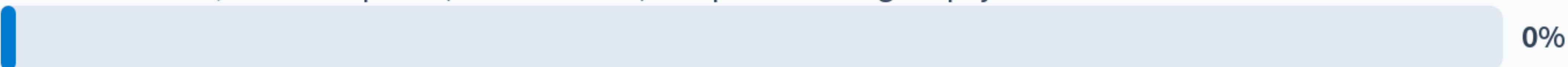
$$\text{var}(F) = 3.24 \cdot \text{var}(T)$$

A company has a weekly pay variance of 100. Different raise structures are considered. In one, they would increase everyone's pay by \$5 per week, in the other they would give a 5% raise to each individual. What happens to the variance?

After the raises, each setting will produce the same pay variance, and this will increase from the current level.



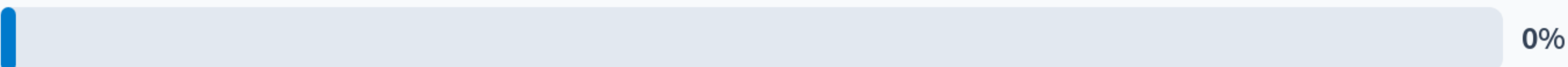
After the raises, the first option (constant raise) will produce a higher pay variance than the second.



After the raises, the second option (percentage raise) will produce a higher pay variance than the first.



After the raises, each setting will produce the same pay variance, and this will decrease from the current level.



Known Distributions

- If $X \sim \text{Bern}(p)$ then

$$E[X] = p \text{ and } \text{var}(X) = p(1 - p)$$

- If $X \sim \text{Geo}(p)$ then

$$E[X] = \frac{1}{p} \text{ and } \text{var}(X) = \frac{1 - p}{p^2}$$